

Exploration Lab - Diophantine Approximations

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Abstract

This paper explores the optimality of finding the best Diophantine approximations. We utilized the geometric structure of Farey tessellations alongside recursive formulas of continued fractions to find an accurate approximation and mapped it using geodesics. Through this combined approach, we demonstrated the use of matrices to uncover each geodesic, leading one closer to the exact value. Finally, using known recursive properties of continued fractions, we define a best approximation.

Continued Fractions

For an irrational number α with continued fraction expansion

$$\alpha = 1 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

and,

$$\alpha = [a_0; a_1, a_2, a_3, \dots]$$

let $\frac{P_n}{Q_n}$ denote the n -th convergent.

$$\begin{aligned} P_1 &= a_1, & Q_1 &= 1 \\ P_2 &= a_1 a_2 + 1, & Q_2 &= a_1 \\ P_n &= a_n P_{n-1} + P_{n-2}, & Q_n &= a_n Q_{n-1} + Q_{n-2}. \end{aligned}$$

Proof

Say

$$\frac{P_n}{Q_n} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \dots}}}$$

We can use the idea of replacement to manipulate the different convergents to obtain the recursive formulas. So,

$$P_{n+1} = (a_n + \frac{1}{a_{n+1}})P_{n-1} + P_{n-2}$$

Similarly,

$$Q_{n+1} = (a_n + \frac{1}{a_{n+1}})Q_{n-1} + Q_{n-2}$$

Now we can represent $\frac{P_{n+1}}{Q_{n+1}}$ as

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{(a_n + \frac{1}{a_{n+1}})P_{n-1} + P_{n-2}}{(a_n + \frac{1}{a_{n+1}})Q_{n-1} + Q_{n-2}}$$

Distributing gives us

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{P_{n-1}a_n + \frac{P_{n-1}}{a_{n+1}} + P_{n-2}}{Q_{n-1}a_n + \frac{Q_{n-1}}{a_{n+1}} + Q_{n-2}}$$

Then,

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{\frac{a_{n+1}P_{n-1}a_n + P_{n-1} + P_{n-2}a_{n+1}}{a_{n+1}}}{\frac{a_{n+1}Q_{n-1}a_n + Q_{n-1} + Q_{n-2}a_{n+1}}{a_{n+1}}}$$

Simplifying further gives,

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{a_{n+1}a_nP_{n-1} + P_{n-1} + P_{n-2}a_{n+1}}{a_{n+1}a_nQ_{n-1} + Q_{n-1} + Q_{n-2}a_{n+1}}$$

Factoring by grouping gives us,

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{a_{n+1}(a_nP_{n-1} + P_{n-2}) + P_{n-1}}{a_{n+1}(a_nQ_{n-1} + Q_{n-2}) + Q_{n-1}}$$

Then, by way of substitution

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{a_{n+1}(P_n) + P_{n-1}}{a_{n+1}(Q_n) + Q_{n-1}}$$

These are the recursive formula to calculate successive convergents.

For example,

$$\sqrt{2}$$

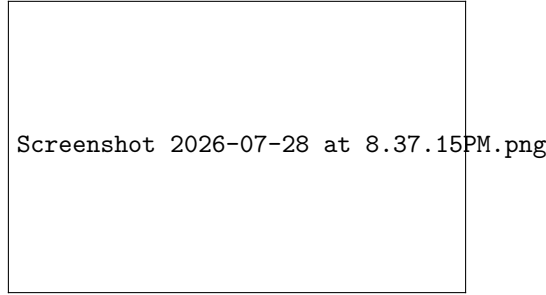


Figure 1: Successive Convergents of $\sqrt{2}$

can be denoted as $[1; 2, 2, 2, \dots]$ where the components of the convergents can be written and calculated as follows,

$$\begin{aligned} P_1 &= 1, & Q_1 &= 1 \\ P_2 &= 1(2) + 1 = 3, & Q_2 &= 2 \\ P_3 &= 2(3) + 1 = 7, & Q_3 &= 2(2) + 1 = 5. \\ P_4 &= 2(7) + 3 = 17, & Q_4 &= 2(5) + 2 = 12. \end{aligned}$$

so the convergents are

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \dots$$

Intuitively, we can see that as the convergents go on, the error (distance between convergent to what we are approximating; α), gets smaller. Another key thing to note is that adjacent approximations oscillate above and below the line $y = \sqrt{2}$.

A key observation: the determinant of the matrix two adjacent convergents is ± 1 . Take the convergents of $\sqrt{2}$ for example,

$$\begin{aligned} M_1 &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} & \det(M_1) &= 3(1) - 2(1) = 1 \\ M_2 &= \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix} & \det(M_2) &= 7(2) - 3(5) = -1 \dots \end{aligned}$$

The error can be represented as

$$\left| \alpha - \frac{P_n}{Q_n} \right|.$$

Theorem

Successive convergents of a continued fraction approximation oscillate above and below the value being approximated. Or more formally,

$$\frac{P_n}{Q_n} > \alpha \quad \text{when } n \text{ is even}$$

$$\frac{P_n}{Q_n} < \alpha \quad \text{when } n \text{ is odd}$$

Proof

Let

$$x = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}$$

Then allow the truncated version, x_{n+1} to equal

$$x_{n+1} = a_2 + \frac{1}{a_3}$$

Then

$$x - a_1 = \frac{1}{x_{n+1}}$$

$$x_{n+1} = \frac{1}{x - a_1}$$

Also,

$$x = \frac{x_{n+1}P_n + P_{n-1}}{x_{n+1}Q_n + Q_{n-1}}$$

We can substitute the previous equation into the one before to obtain

$$x_{n+1} = \frac{1}{\frac{x_{n+1}P_n + P_{n-1}}{x_{n+1}Q_n + Q_{n-1}} - a_1}$$

Distributing gives us,

$$x_{n+1} = \frac{1}{\frac{x_{n+1}P_n + P_{n-1} - a_1x_{n+1}Q_n - a_1Q_{n-1}}{x_{n+1}Q_n + Q_{n-1}}}$$

We can simplify this to,

$$x_{n+1} = \frac{x_{n+1}Q_n + Q_{n-1}}{x_{n+1}P_n + P_{n-1} - a_1x_{n+1}Q_n - a_1Q_{n-1}}$$

Assume that n is even We can represent the distance from x to a convergent as,

$$x - \frac{P_n}{Q_n} = \frac{x_{n+1}P_n + P_{n-1}}{x_{n+1}Q_n + Q_{n-1}} - \frac{P_n}{Q_n}$$

Subtracting these fractions gives us,

$$x - \frac{P_n}{Q_n} = \frac{Q_n(x_{n+1}P_n + P_{n-1}) - P_n(x_{n+1}Q_n + Q_{n-1})}{Q_n(x_{n+1}Q_n + Q_{n-1})}$$

By the earlier lemma,

$$x - \frac{P_n}{Q_n} = \frac{(-1)^n}{Q_n(x_{n+1}Q_n + Q_{n-1})}$$

We know that n is even and the denominators are always positive. A fraction with a positive numerator and denominator is positive. $\frac{+}{+} = +$. Hence $x - \frac{P_n}{Q_n} > 0$

Similarly, we can apply the same logic to $x - \frac{P_n}{Q_n}$ to show that $x - \frac{P_{n+1}}{Q_{n+1}} < 0$

$$|\alpha - \frac{P_n}{Q_n}| \leq |\frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n}| = \frac{1}{Q_n Q_{n+1}}$$

Proof

Lemma:

$$P_{n+1}Q_n - P_nQ_{n+1} = (-1)^n$$

Let $D_n = P_{n+1}Q_n - P_nQ_{n+1}$. We want to prove that D_n is always either 1 or -1, and that its sign alternates. The numerators and denominators of consecutive convergents satisfy these recursive formulas:

$$P_{n+1} = a_{n+1}P_n + P_{n-1}, \quad Q_{n+1} = a_{n+1}Q_n + Q_{n-1}.$$

Start with D_n : $D_n = P_{n+1}Q_n - P_nQ_{n+1}$ substitute P_{n+1} and Q_{n+1} using the recursive formulas. This gives

$$D_n = (a_{n+1}P_n + P_{n-1})Q_n - P_n(a_{n+1}Q_n + Q_{n-1})$$

Distributing gives

$$D_n = a_{n+1}P_nQ_n + P_{n-1}Q_n - a_{n+1}P_nQ_n - P_nQ_{n-1}$$

Since α lies between the consecutive convergents $\frac{P_n}{Q_n}$ and $\frac{P_{n+1}}{Q_{n+1}}$, we know that

$$|\alpha - \frac{P_n}{Q_n}| \leq |\frac{P_{n+1}}{Q_{n+1}} - \frac{P_n}{Q_n}|.$$

Using the lemma, we can prove the notion that $|\alpha - \frac{P_n}{Q_n}| \leq \frac{1}{Q_n Q_{n+1}} \leq \frac{1}{Q_n^2}$

Let $\alpha = [a_1, a_2, \dots]$ be the simple continued fraction of α . If a fraction $\frac{P}{Q}$ gives a better approximation to α than the k th convergent $\frac{P_k}{Q_k}$ then $Q > Q_k$

Proof

Let $t = [a_{k+1}, a_{k+2}, \dots]$ Since all continued fraction terms are positive, $t > 1$. The recursive formulas to calculate convergents give us

$$\alpha = \frac{tP_k + P_{k+1}}{tQ_k + Q_{k+1}}$$

Let

$$D = tQ_k + Q_{k+1} > 0$$

Then

$$\alpha = \frac{tP_k + P_{k-1}}{D}$$

Calculating the error of the k^{th} convergent gives us

$$Q_k\alpha - P_k = Q_k \frac{tP_k + P_{k-1}}{D} - P_k = \frac{Q_k tP_k + Q_k P_{k-1} - P_k(tQ_k + Q_{k-1})}{D} = \frac{Q_k P_{k-1} - P_k Q_{k-1}}{D}$$

We have proved the identity that consecutive convergents satisfy

$$P_{n+1}Q_n - P_nQ_{n+1} = (-1)^n$$

Then,

$$|Q_k\alpha - P_k| = \frac{1}{D}$$

Because $P_{n+1}Q_n - P_nQ_{n+1} = \pm 1$, (P_k, Q_k) and (P_{k-1}, Q_{k-1}) can be uniquely written as an integer combo of two consecutive convergents.

Therefore, $\exists r, s \in \mathbb{Z}$ such that

$$P = rP_k + sP_{k-1}, \quad Q = rQ_k + sQ_{k-1}$$

So

$$Q\alpha - P = r(Q_k\alpha - P_k) + s(Q_{k-1}\alpha - P_{k-1})$$

We also know that

$$Q_{k-1}\alpha - P_{k-1} = (r - st)(Q_k\alpha - P_k)$$

Therefore,

$$|Q_{k-1}\alpha - P_{k-1}| = |r - st||Q_k\alpha - P_k|$$

Suppose, for the sake of contradiction, $Q \leq Q_k$. Then if $s = 0$, $Q = rQ_k$. Since $0 < Q \leq Q_k$, we must have $r=1$. This leads to $(P, Q) = (P_k, Q_k)$ so they are the same fraction.

Now suppose $s \neq 0$:

Case 1: $s > 0$

Since

$$Q = rQ_k + sQ_{k-1}$$

we cannot have $r \geq 1$ because then $Q \geq Q_k + sQ_{k-1} > Q_k$.

Thus, $r \leq 0$. Since $t > 1$, $r - st < -1$. Therefore, $|r - st| > 1$

Case 2: $s < 0$

Assume $s = -u$ for some $u \in \mathbb{Z}$ where $u > 0$ then $Q = rQ_k - uQ_{k-1} > 0$. This forces $r \geq 1$. Therefore $r - st = rtut > 1$ so again $|r - st| > 1$ so unless $\frac{P}{Q} = \frac{P_k}{Q_k}$, $|r - st| \geq 1$. Therefore, $|Q\alpha - P| \geq |Q_k\alpha - P_k|$.

Since $|\alpha - \frac{P}{Q}| = \frac{|Q_k\alpha - P_k|}{Q}$, we have $|\alpha - \frac{P}{Q}| \geq \frac{|Q_k\alpha - P_k|}{Q}$.

Because $Q \leq Q_k$,

$$\frac{|Q_k\alpha - P_k|}{Q} \leq \frac{|Q_k\alpha - P_k|}{Q_k}$$

But

$$\frac{|Q_k\alpha - P_k|}{Q_k} = |\alpha - \frac{P_k}{Q_k}|$$

Hence $|\alpha - \frac{P}{Q}| \geq |\alpha - \frac{P_k}{Q_k}|$ so no fraction with denominator $Q \leq Q_k$ can be a better approximation than the k^{th} convergent.

□

Farey Tessellations

$$F_1 : \frac{0}{1}, \frac{1}{1} \tag{1}$$

$$F_2 : \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \tag{2}$$

$$F_3 : \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \tag{3}$$

$$F_4 : \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1} \tag{4}$$

Let the k th Farey Fraction of order n be $\frac{F_{kn}}{G_{kn}}$.

Note: Consecutive Farey Fractions written as a matrix will always create a determinant of 1:

$$\begin{bmatrix} F_{n+1} & F_n \\ G_{n+1} & G_n \end{bmatrix} = F_{n+1}G_n - F_nG_{n+1} = 1$$

Define the upper half plane to be

$$\mathbb{H} = \{z = x + iy \mid y > 0\} \cup \infty.$$

The functions that act on the upper half-plane have the form of a fractional linear function:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times z = \frac{az + b}{cz + d}$$

Definition: Farey tessellations are patterns of curved triangles drawn in the upper half of the complex plane

Definition: Geodesics are vertical straight lines or curved semicircles. Triangles are formed by 3 geodesics. Geodesics are formed between 2 consecutive Farey fractions that have determinant 1.

We saw that applying matrices with determinant 1 that mapped imaginary values of z to fractional linear function form created a geodesic with left endpoint $\frac{b}{d}$ and right endpoint $\frac{a}{c}$.

*insert numerical examples here

Proof: Since we view geodesics as lines with endpoints $z = 0$ and $z = \infty$, we can observe the following:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times 0 = \frac{b}{d}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \infty = \frac{a}{c}.$$

We define a Farey triangle as an ideal triangle in the upper-half plane whose vertices are given by distinct Farey fractions or infinity.

Note: These triangles are bounded either by three geodesics or by a single geodesic and two vertical lines that extend to infinity.

*insets examples

The primary limitation of this approach is the difficulty in tracing the trajectory of the irrational geodesic upon entering a Farey triangle. This challenge motivated our approach to systematically remove the outer layers of geodesics.

Matrix: D

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{z}$$

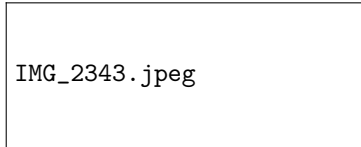


Figure 2: Action of Matrix D

This matrix successfully removed the outer layer of geodesics by reflecting across the real axis. However, the imaginary part of the geodesics becomes negative, which is not defined in the upper half plane.

Therefore, we are motivated to find a new matrix with the same property of removing the outer layer of geodesics. This maintains all positive imaginary coordinates.

Additionally, note that the determinant of D is -1 , not 1 .

Matrix: S

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times z \mapsto \frac{-1}{z}$$

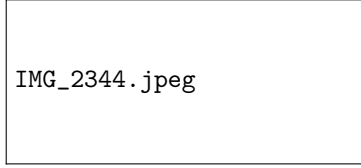


Figure 3: Action of Matrix S

The matrix S, which has a determinant 1, maps the geodesic with one endpoint at 0 and a nonzero endpoint p to the line from $-\frac{1}{p} \rightarrow \infty$. This successfully removes the outer geodesic layer and allows us to view more accurate approximations of an irrational geodesic. The limitation with S is that it only acts on geodesics with one endpoint at zero. This means that we will need another matrix to shift the geodesics left and right in order to act on geodesics with two nonzero endpoints.

Matrix: T and T^{-1}

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mapsto z + 1$$

$$T^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \mapsto z - 1$$

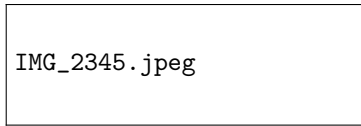


Figure 4: Action of Inverse of T

The fractional linear function form of T lets us see that applying T shifts geodesics exactly one unit left, and applying T^{-1} shifts geodesics exactly one unit right. Since the determinant of T is 1, we now have all the matrices necessary to complete our goal of peeling away geodesics.

Applying Matrices Geometrically and Algebraically

Let's take the numerical example of $\sqrt{6}$ from earlier. Below, find the image of the geodesic for $\sqrt{6}$ graphed with desmos.

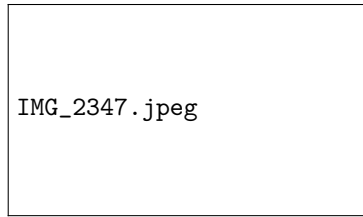


Figure 5: Action of Inverse T

The first geodesic that intersects our irrational geodesic falls in between 2 and 3, so we want to shift the graph two units to the left so we get endpoints 0 and 1. Thus, we apply T^{-2} . Now, we can remove the outer layer of geodesics.

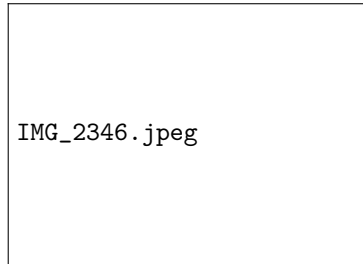


Figure 6: Action of Inverse T

graph of

$$(T^{-2}S)$$

Consequently, this isolates a closer rational approximation to the irrational point.

Let e be the next element in the continued fraction.

$$\begin{pmatrix} P_k & P_{k+1} \\ Q_k & Q_{k+1} \end{pmatrix} \rightarrow \begin{pmatrix} P_{k+1} & P_{k+2} \\ Q_{k+1} & Q_{k+2} \end{pmatrix}$$

The periodic continued fraction:

$$\sqrt{6} = [2; \overline{2, 4}]$$

has a string;

$$[\text{LLRRLLLLRRLLLL} \dots]$$

That is, a sequence of the geodesic line across the Farey tessellation. The alternating lengths of the letter represent where the irrational crosses in each geodesic.

$$\text{Then Let } S = [(T^2S)(T^{-2}S)(T^4S)(T^{-2}S) \dots]$$

Optimality of Continued Fractions

Finally, we must prove that continued fractions are the best sequence of fractional approximations for any irrational number. Let $\frac{p}{q}$ be any rational fraction, and let $\frac{P_k}{Q_k}$ be the kth convergent. We have:

$$|\alpha - \frac{p}{q}| < |\alpha - \frac{P_k}{Q_k}| \implies q > Q_k$$

Proof: Assume for contradiction, that $q \leq Q_k$. By definition, this means that they are part of the same Farey sequence. We have that if $\frac{a}{b}$ is adjacent to $\frac{c}{d}$, then in the next sequence, the fractions $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{a+b}{c+d}$ are all adjacent.

The continued fraction between $\frac{P_k}{Q_k}$ and $\frac{P_{k+1}}{Q_{k+1}}$ is $\frac{P_k + P_{k+1}}{Q_k + Q_{k+1}}$.

$$\frac{P_k + P_{k+1}}{Q_k + Q_{k+1}} > \frac{p}{q} \implies q > Q_k$$

Thus, we have a contradiction, and $q > Q_k$.